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Representations of votes based on pairwise information: Monotonicity versus consistency

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ABSTRACT

Representations of votes, such as the scorix, the votrix or the beatpath matrix, are common tools in social choice theory. They gather the most relevant information given by the voters and are key elements in most of the existing ranking rules. The goal of this paper is twofold. First, the most relevant representations of votes based on pairwise information are revisited. Second, three properties (consistency, pseudo-monotonicity and monotonicity) of these representations of votes are introduced. Although similar notions to that of consistency have been analysed in social choice theory since the eighteenth century, pseudo-monotonicity and monotonicity have only called the attention of the research community in recent years.

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1. Introduction

The aggregation of the rankings expressed by several voters into a unique ranking is a classical problem in social choice theory. All these provided rankings are usually compressed into a representation of votes gathering the most significant information. For instance, the Borda count [2], probably the most well-known ranking rule, reduces the rankings given by the voters to the scoring matrix [9] (from now on referred to as the scorix [20]), which is a matrix where the element at the ith row and jth column equals the number of times that the ith candidate is ranked at the jth position in the rankings given by the voters. Besides the Borda count, all other scoring ranking rules [11,16,17,29] also reduce the rankings given by the voters to the scorix.

Based on the ideas of Condorcet [4], another representation of votes that gathers a completely different type of information is also commonly used in social choice theory: the voting matrix [30] (from now on referred to as the votrix [19]). The votrix is a matrix where the element at the ith row and jth column equals the number of times that the ith candidate is preferred to the jth candidate in the rankings given by the voters. This type of representation of votes where candidates are head-to-head compared with each other is said to be based on pairwise information. Many other representations of votes based on pairwise information, such as the beatpath matrix [25] and the votex [19], have also been considered in the field of social choice theory.

Here, we discuss three different potential properties of representations of votes based on pairwise information: consistency, pseudo-monotonicity and monotonicity. On the one hand, consistency is a natural property based on the ideas of Condorcet that has been studied under different names since the early days of social choice theory [4,30]. On the other

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hand, pseudo-monotonicity and monotonicity are two properties that have been long-time disregarded in the field and are recently attracting the attention of the scientific community [19,22].

The (mono)metric rationalisation of ranking rules [1,7,21] is a branch of social choice theory that aims to characterize ranking rules in terms of the search for an appropriate closest consensus state. Historically, only unanimity [14] and the presence of a Condorcet winner [5] have been considered suitable for this purpose. Nevertheless, we have recently advocated for the use of new consensus states, such as monotonicity of the scorix, monotonicity of the votrix and monotonicity of the votex [19,20]. Obviously, any new property of a representation of votes raises the question whether it could serve as a meaningful consensus state for the (mono)metric rationalisation of ranking rules.

First, the most well-known representations of votes based on pairwise information are recalled in Section 2. Second, the notions of consistency, pseudo-monotonicity and monotonicity of a representation of votes are analysed in Section 3. A general discussion on the relations between monotonicity and consistency of the most well-known representations of votes based on pairwise information is addressed in Section 4. Finally, some conclusions and open problems are discussed in Section 5.

2. Representations of votes

In this section, the notion of a representation of votes (based on pairwise information) is introduced. In addition, several existing constructs, such as the α-majority matrix, the beatpath matrix, the votrix and the votex, are shown to fit into this definition.

2.1. Definition

We consider a set $\mathscr{C} = \{a_1, \ldots, a_k\}$ of k candidates and r voters are asked to express their preferences on the set of candidates. Each voter expresses his/her preferences in the form of a strict linear order relation or ranking $\succ_j$ on $\mathscr{C}$, i.e. the asymmetric part of a total order relation $\succeq_j$ on $\mathscr{C}$. The set of all rankings on $\mathscr{C}$ is denoted by $\mathscr{L}(\mathscr{C})$. The set of all couples of two different candidates is denoted by $\mathscr{C}^2_{\neq}$.

The list of r rankings given by the voters is called a profile of rankings and is denoted by $\mathscr{R} = (\succ_j)_{j=1}^r$. The set of all possible permutations of the elements of a set S is denoted by $\Pi(S)$. For any $\sigma \in \Pi(\{1, \ldots, r\})$, $\mathscr{R}_\sigma$ represents the profile $(\succ_j)_{j=\sigma(1)}^{\sigma(r)}$ obtained by permuting the order of the voters. For any $\tau \in \Pi(\{1, \ldots, k\})$, $\tau(\mathscr{R})$ represents the profile $(\tau(\succ_j))_{j=1}^r$ obtained by permuting the order of the candidates.

In order to look for the ranking that best represents the characteristics of the preferences given by the voters, it is common to reduce the profile of rankings into a representation of votes. A representation of votes needs to satisfy three basic properties: anonymity (reassigning the rankings over the voters does not change the representation of votes), neutrality (if some permutation of candidates is applied to each voter's ranking, the same permutation should be observed in the representation of votes) and unanimity (if every voter provides the same ranking, then the representation of votes reflects the same natural ranking).

Definition 1. Let $\mathscr{C}$ be a set of k candidates, r be the number of voters and $n \in \mathbb{N}$. A function $T : \mathscr{L}(\mathscr{C})^r \times \mathscr{C}^2_{\neq} \to \mathbb{R}^n$ is called a representation of votes (based on pairwise information) if it satisfies the following properties:

(i) **Anonymity:** for any $\mathscr{R} \in \mathscr{L}(\mathscr{C})^r$, any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$ and any $\sigma \in \Pi(\{1, \ldots, r\})$, it holds that

$$T\big(\mathscr{R}, (a_{i_1}, a_{i_2})\big) = T\big(\mathscr{R}_\sigma, (a_{i_1}, a_{i_2})\big).$$

(ii) **Neutrality:** for any $\mathscr{R} \in \mathscr{L}(\mathscr{C})^r$, any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$ and any $\tau \in \Pi(\{1, \ldots, k\})$, it holds that

$$T\big(\mathscr{R}, (a_{i_1}, a_{i_2})\big) = T\big(\tau(\mathscr{R}), (a_{\tau(i_1)}, a_{\tau(i_2)})\big).$$

(iii) **Unanimity:** there exists an intuitive strict order relation $\rhd_T$ on the range $T\big(\mathscr{L}(\mathscr{C})^r \times \mathscr{C}^2_{\neq}\big)$ (with corresponding reflexive closure denoted by $\trianglerighteq_T$) such that, for any $\succ \in \mathscr{L}(\mathscr{C})$, any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$ satisfying that $a_{i_1} \succ a_{i_2}$ and any $a_{i_3} \in \mathscr{C} \setminus \{a_{i_1}, a_{i_2}\}$, it holds that

$$T\big(\succ^r, (a_{i_1}, a_{i_2})\big) \rhd_T T\big(\succ^r, (a_{i_2}, a_{i_1})\big),$$
$$T\big(\succ^r, (a_{i_1}, a_{i_3})\big) \trianglerighteq_T T\big(\succ^r, (a_{i_2}, a_{i_3})\big),$$
$$T\big(\succ^r, (a_{i_3}, a_{i_2})\big) \trianglerighteq_T T\big(\succ^r, (a_{i_3}, a_{i_1})\big).$$

As the profile $\mathscr{R}$ of rankings is usually fixed, the restriction of the representation of votes to $\{\mathscr{R}\} \times \mathscr{C}^2_{\neq} \subseteq \mathscr{L}(\mathscr{C})^r \times \mathscr{C}^2_{\neq}$ is often considered.

Table 1
Frequency of the rankings on $\mathscr{C}$ expressed by 14 voters.

$\#\succ_i$	Rankings on $\mathscr{C}$
6	$c\succ b\succ a\succ d$
5	$a\succ d\succ b\succ c$
3	$b\succ a\succ d\succ c$

Definition 2. Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n\in\mathbb{N}$ and $T:\mathcal{L}(\mathscr{C})^r\times\mathscr{C}^2_{\neq}\to\mathbb{R}^n$ be a representation of votes. The function $T_{\mathscr{R}}:\mathscr{C}^2_{\neq}\to\mathbb{R}^n$, defined as

$$T_{\mathscr{R}}(a_{i_1},a_{i_2})=T\big(\mathscr{R},(a_{i_1},a_{i_2})\big),$$

for any $(a_{i_1},a_{i_2})\in\mathscr{C}^2_{\neq}$, is called the (instance of the)[1] representation of votes induced by the profile.

For the instance $T_{\mathscr{R}}$ of the representation of votes T induced by the profile $\mathscr{R}$, $T_{\mathscr{R}}(a_{i_1},a_{i_2})$ is referred to as the support of candidate a_{i_1} over candidate a_{i_2}.

Although every profile of rankings induces a unique instance of the representation of votes, it is usually not possible to recover the profile of rankings from the representation of votes. It might be even the case that different profiles of rankings induce the same instance of the representation of votes.

2.2. Relevant representations of votes

In the following, we mention the most prominent representations of votes (based on pairwise information).

The simple majority rule [10,13,15,27] is one of the most widespread and studied concepts in social choice theory. A candidate a_{i_1} defeats another candidate a_{i_2} by simple majority if the number of voters who prefer a_{i_1} to a_{i_2} is greater than the number of voters who prefer a_{i_2} to a_{i_1}. A closely related concept is that of the (weak) absolute majority, where a candidate a_{i_1} defeats another candidate a_{i_2} by absolute majority if the number of voters who prefer a_{i_1} to a_{i_2} is greater than (or equal to) half of the number of voters. As in the setting of this paper each voter provides his/her preferences in the form of a ranking on the set of candidates, both simple and absolute majority coincide.

Unfortunately, both simple and absolute majority might lead to the well-known voting paradox [4] (also known as Condorcet's paradox), where a candidate a_{i_1} defeats a second candidate a_{i_2}, this candidate a_{i_2} defeats a third different candidate a_{i_3}, which, at the same time, defeats a_{i_1}. In order to avoid such paradox the notion of unanimous majority arises. A candidate a_{i_1} defeats another candidate a_{i_2} by unanimous majority if every voter prefers a_{i_1} to a_{i_2}. Nevertheless, although unanimous majority avoids the voting paradox, it obviously is too strong a condition and almost never holds in an election.

Another type of majority that lies in between the notions of absolute majority and unanimous majority are qualified majorities [8]. Qualified majorities require the number of voters who prefer a_{i_1} to a_{i_2} to be greater than or equal to certain quota αr (where r equals the number of voters and $\alpha\in[0.5,1]$), fixed before the election. Note that $\alpha=0.5$ corresponds to weak absolute majority and $\alpha=1$ corresponds to unanimous majority.

In the same way that qualified majorities lie in between absolute majority and unanimous majority, there exists another type of majority (majority based on differences of votes) that lies in between simple majority and unanimous majority. Qualified majorities and majorities based on differences of votes become equivalent in the setting of this paper, where each voter provides his/her preferences in the form of a ranking on the set of candidates. For more details on all these types of majority, we refer to [18].

Qualified majorities (and all the other types of majority) lead to a natural representation of votes.

Definition 3. Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. For any $\alpha\in[\frac{1}{2},1]$, the representation of votes $M^\alpha:\mathcal{L}(\mathscr{C})^r\times\mathscr{C}^2_{\neq}\to\{0,1\}$ defined as

$$M^\alpha\big(\mathscr{R},(a_{i_1},a_{i_2})\big)=\begin{cases}0, & \text{if } \#\{j\in\{1,\dots,r\}\mid a_{i_1}\succ_j a_{i_2}\}<r\cdot\alpha,\\ 1, & \text{if } \#\{j\in\{1,\dots,r\}\mid a_{i_1}\succ_j a_{i_2}\}\geq r\cdot\alpha,\end{cases}$$

for any $\mathscr{R}\in\mathcal{L}(\mathscr{C})^r$ and any $(a_{i_1},a_{i_2})\in\mathscr{C}^2_{\neq}$, is called the α-majority matrix.[2]

Example 1. Let us consider a set of four candidates $\mathscr{C}=\{a,b,c,d\}$ and a profile of rankings $\mathscr{R}=(\succ_i)_{i=1}^{14}$ provided by fourteen voters. These rankings are shown in Table 1.

[1] When the representation of votes is known by a name, say votrix, the phrase 'instance of the' is omitted, i.e. we refer to 'the instance of the representation of votes induced by the profile' or to 'the votrix induced by the profile', but we do not refer to 'the instance of the votrix induced by the profile'.

[2] The α-majority matrix is directly represented in the form of a matrix of zeros and ones or in the form of a binary relation.

For $\alpha = 0.5$, we can see that candidate b is preferred to candidate a by at least $\alpha r = 7$ voters. Therefore, b is preferred to a by a 0.5-majority (weak absolute majority). The 0.5-majority matrix induced by the profile $\mathscr{R}$ is

$$M^{0.5}_{\mathscr{R}} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For $\alpha = 1$, we can see that candidate a is preferred to candidate c by at least $\alpha r = 14$ voters. Therefore, a is preferred to c by a 1-majority (unanimous majority). The 1-majority matrix induced by the profile $\mathscr{R}$ is

$$M^{1}_{\mathscr{R}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As previously discussed, the voting paradox is a big drawback for the α-majority rule (with $\alpha < 1$). Several ways of avoiding this paradox have been proposed, the request of a unanimous majority being the oldest one. It is also common to avoid the voting paradox by obtaining choice sets [3], such as the Smith set [12,28] and the Schwarz set [26].

Here, we focus on a third option, namely the use of Schulze's beatpaths [25], which leads to another natural representation of votes: the beatpath matrix. Recall that a path from a candidate a_{i_1} to a candidate a_{i_2}, denoted by $a_{i_1} \to a_{i_2}$, is a list of couples $\big((a^i, a^{i+1})\big)_{i=1}^{n}$ such that $a^1 = a_{i_1}$, $a^{n+1} = a_{i_2}$ and $(a^1, \ldots, a^{n+1})$ are pairwisely different. The weakest link of a path $\big((a^i, a^{i+1})\big)_{i=1}^{n}$ from a_{i_1} to a_{i_2} is the couple (a^i, a^{i+1}) such that the number of voters preferring a^i to a^{i+1} (called the strength of the weakest link) is the minimum among all the couples in $\big((a^i, a^{i+1})\big)_{i=1}^{n}$. A path from a_{i_1} to a_{i_2} that maximizes the strength of the weakest link is called a beatpath[3] from a_{i_1} to a_{i_2}.

Definition 4. Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. The representation of votes $B : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}^2_{\neq} \to \{0, 1, \ldots, r\}$ defined as

$$B\big(\mathscr{R}, (a_{i_1}, a_{i_2})\big) = \max_{a_{i_1} \to a_{i_2}} \min_{(a^i, a^{i+1}) \in \to} \#\{j \in \{1, \ldots, r\} \mid a^i \succ_j a^{i+1}\},$$

for any $\mathscr{R} \in \mathcal{L}(\mathscr{C})^r$ and any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$, is called the beatpath matrix.[4]

Remark 1. In Schulze's original proposal [25], the strength of the weakest link is given by the couple formed by the number of voters preferring a_{i_1} to a_{i_2} and the number of voters preferring a_{i_2} to a_{i_1}. However, as in the setting of this paper each voter provides his/her preferences in the form of a ranking on the set of candidates, the first element of the couple determines the second one.

Example 2. Let us continue with the set of candidates and the profile of rankings given in Example 1. We can see that there are five different paths from a candidate a_{i_1} to a candidate a_{i_2}. For instance, for going from a to b, there are the following five possible paths:

$$\begin{aligned} \big((a,b)\big) &\text{ leading to } \min(5) = 5, \\ \big((a,c),(c,b)\big) &\text{ leading to } \min(8,6) = 6, \\ \big((a,d),(d,b)\big) &\text{ leading to } \min(14,5) = 5, \\ \big((a,c),(c,d),(d,b)\big) &\text{ leading to } \min(8,6,5) = 5, \\ \big((a,d),(d,c),(c,b)\big) &\text{ leading to } \min(14,8,6) = 6. \end{aligned}$$

The paths maximizing the strength of the weakest link are ((*a, c*), (*c, b*)) and ((*a, d*), (*d, c*), (*c, b*)) with a value of six. Therefore, the element of the beatpath matrix corresponding to (*a, b*) equals six. The beatpath matrix induced by the profile $\mathscr{R}$ is

$$B_{\mathscr{R}} = \begin{pmatrix} - & 6 & 8 & 14 \\ 9 & - & 8 & 9 \\ 6 & 6 & - & 6 \\ 6 & 6 & 8 & - \end{pmatrix}.$$

Probably the most common representation of votes is the votrix [19] (also known as voting matrix [30]). The votrix gathers the number of times that a candidate is preferred to another candidate (called the strength of support of the first candidate over the second candidate).

[3] Schulze [25] actually allows candidates to appear several times in the path. Nevertheless, both approaches are equivalent when maximizing the strength of the weakest link.

[4] The beatpath matrix is directly represented in the form of a matrix with the elements at the diagonal left undefined.

Definition 5. Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. The representation of votes $V : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \{0, 1, \ldots, r\}$ defined as

$$V\big(\mathscr{R}, (a_{i_1}, a_{i_2})\big) = \#\{j \in \{1, \ldots, r\} \mid a_{i_1} \succ_j a_{i_2}\},$$

for any $\mathscr{R} \in \mathcal{L}(\mathscr{C})^r$ and any $(a_{i_1}, a_{i_2}) \in \mathscr{C}_{\neq}^2$, is called the votrix.[5]

Example 3. Let us continue with the set of candidates and the profile of rankings given in Example 1. We can see that the number of voters preferring a to b is five. Therefore, the element of the votrix corresponding to (a, b) equals five. The votrix induced by the profile $\mathscr{R}$ is

$$V_{\mathscr{R}} = \begin{pmatrix} 0 & 5 & 8 & 14 \\ 9 & 0 & 8 & 9 \\ 6 & 6 & 0 & 6 \\ 0 & 5 & 8 & 0 \end{pmatrix}.$$

A common criticism against contracting the profile of rankings into the votrix, in particular, and the methods based on pairwise information, in general, is that they do not exploit the fact that each voter expresses a ranking on the set of candidates. As an example of such received criticism, we refer to a brief reflection by Saari [24]: "the combination of the pairwise vote with the Condorcet terms loses the crucial fact that voters have transitive preferences. [...] An equally surprising assertion is that rather than being the standard, the Condorcet winner must be held suspect". Pérez-Fernández et al. [19] proposed a new representation of votes that, although it is based on pairwise information, gathers the positional information given by the voters when providing a ranking. This representation of votes is called the votex.

Definition 6. Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. The representation of votes $W : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \{0, 1, \ldots, r\}^{2k-2}$ defined as

$$W\big(\mathscr{R}, (a_{i_1}, a_{i_2})\big) = \big(n_{1-k}(a_{i_1}, a_{i_2}), \ldots, n_{-1}(a_{i_1}, a_{i_2}), n_1(a_{i_1}, a_{i_2}), \ldots, n_{k-1}(a_{i_1}, a_{i_2})\big),$$

for any $\mathscr{R} \in \mathcal{L}(\mathscr{C})^r$ and any $(a_{i_1}, a_{i_2}) \in \mathscr{C}_{\neq}^2$, where[6]

$$n_i(a_{i_1}, a_{i_2}) = \#\{j \in \{1, \ldots, r\} \mid P_j(a_{i_2}) - P_j(a_{i_1}) = i\},$$

for any $i \in \{1-k, \ldots, -1, 1, \ldots, k-1\}$, is called the votex.

Example 4. Let us continue with the set of candidates and the profile of rankings given in Example 1. We can see that it holds that $W_{\mathscr{R}}(a, c) = (0, 6, 0, 0, 3, 5)$, since, in the ranking $c \succ b \succ a \succ d$, c is preferred to a and there is one candidate in between them; in the ranking $a \succ d \succ b \succ c$, a is preferred to c and there are two candidates in between them; in the ranking $b \succ a \succ d \succ c$, a is preferred to c and there is one candidate in between them. The votex induced by the profile $\mathscr{R}$ is

$$\begin{aligned}
W_{\mathscr{R}}(a, b) &= (0, 0, 9, 0, 5, 0), & W_{\mathscr{R}}(c, a) &= (5, 3, 0, 0, 6, 0), \\
W_{\mathscr{R}}(a, c) &= (0, 6, 0, 0, 3, 5), & W_{\mathscr{R}}(c, b) &= (3, 0, 5, 6, 0, 0), \\
W_{\mathscr{R}}(a, d) &= (0, 0, 0, 14, 0, 0), & W_{\mathscr{R}}(c, d) &= (0, 5, 3, 0, 0, 6), \\
W_{\mathscr{R}}(b, a) &= (0, 5, 0, 9, 0, 0), & W_{\mathscr{R}}(d, a) &= (0, 0, 14, 0, 0, 0), \\
W_{\mathscr{R}}(b, c) &= (0, 0, 6, 5, 0, 3), & W_{\mathscr{R}}(d, b) &= (0, 9, 0, 5, 0, 0), \\
W_{\mathscr{R}}(b, d) &= (0, 0, 5, 0, 9, 0), & W_{\mathscr{R}}(d, c) &= (6, 0, 0, 3, 5, 0).
\end{aligned}$$

Note that the use of the α-majority matrix dates back to the eighteenth century, when Rousseau [23] already encouraged the use of qualified majorities for important decisions: "The more the deliberations are important and serious, the more the opinion that carries should approach unanimity". Contemporaneously, the use of the votrix is known to have been used by Condorcet [4]. However, this use of the votrix was most of the times an intermediate step for the use of the 0.5-majority matrix, and it was not until the twentieth century when the exploitation of all the information gathered by the votrix started attracting the attention of the scientific community [6,22]. The beatpath matrix was born with the intention of avoiding the voting paradox and introducing an interesting voting rule that is proved to satisfy a lot of properties (see the title of [25] for a brief summary of these satisfied properties). Lastly, the votex is a newly introduced representation of votes aiming to exploit both pairwise and positional information at the same time [19].

3. Monotonicity versus consistency

As mentioned in the previous sections, for each representation of votes T, there exists an intuitive strict order relation $\rhd_T$ on the range $T\big(\mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2\big)$. For the α-majority matrix, the beatpath matrix and the votrix, the strict order relation $\rhd_T$

[5] The votrix is directly represented in the form of a matrix.

[6] For any ranking $\succ_j \in \mathscr{R}$ and any $a_{i_1} \in \mathscr{C}$, the position of candidate a_{i_1} in $\succ_j$ is denoted by $P_j(a_{i_1})$.

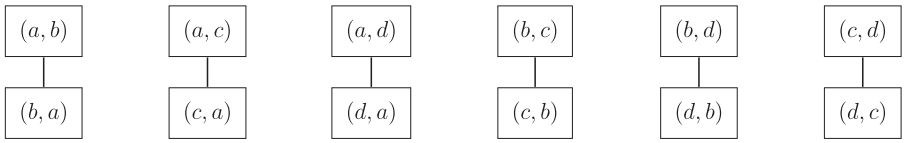

Fig. 1. Hasse diagram of $\sqsupset_{\succ}^{C}$ for the ranking $a \succ b \succ c \succ d$.

is the usual strict order relation $>$ on $\mathbb{R}$; for the votex, the strict order relation $\rhd_W$ is the strict dominance relation between vectors of the same size $2n$ that sum up to r, defined by: $A \rhd_W B$ if

$$\sum_{j=1}^{i} A_j \leq \sum_{j=1}^{i} B_j\,,$$

for any $i \in \{1, \ldots, 2n\}$, and

$$\sum_{j=1}^{n} A_j < \sum_{j=1}^{n} B_j\,.$$

The notions of monotonicity and consistency are defined according to the strict order relation $\rhd_T$ (and the reflexive closure denoted by $\trianglerighteq_T$).

Consistency w.r.t. a ranking $\succ$ on $\mathscr{C}$ means that, if $a_{i_1} \succ a_{i_2}$, then the support of a_{i_1} over a_{i_2} needs to be greater than (or equal to) the support of a_{i_2} over a_{i_1}.

Definition 7. Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$ and $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be a representation of votes. The instance $T_{\mathscr{R}} : \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ of T induced by the profile $\mathscr{R}$ is said to be

(i) (strictly) consistent w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ if, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}_{\neq}^2$, it holds that

$$a_{i_1} \succ a_{i_2} \Rightarrow T_{\mathscr{R}}(a_{i_1}, a_{i_2}) \rhd_T T_{\mathscr{R}}(a_{i_2}, a_{i_1})\,;$$

(ii) weakly consistent w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ if, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}_{\neq}^2$, it holds that

$$a_{i_1} \succ a_{i_2} \Rightarrow T_{\mathscr{R}}(a_{i_1}, a_{i_2}) \trianglerighteq_T T_{\mathscr{R}}(a_{i_2}, a_{i_1})\,.$$

Intuitively, we can see that consistency w.r.t. a ranking $\succ$ on $\mathscr{C}$ means that the instance of the representation of votes induced by the profile is (strictly) decreasing on the diagram defined by the following strict partial order relation.

Definition 8. Let $\mathscr{C}$ be a set of k candidates. A ranking $\succ \in \mathcal{L}(\mathscr{C})$ induces a strict partial order relation $\sqsupset_{\succ}^{C}$ on $\mathscr{C}_{\neq}^2$ such that, for any two $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}_{\neq}^2$, it holds that $(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{C} (a_{j_1}, a_{j_2})$ if

$$(a_{i_1} \succ a_{i_2}) \wedge (a_{i_1} = a_{j_2}) \wedge (a_{i_2} = a_{j_1})\,.$$

In Fig. 1, a graphical representation of $\sqsupset_{\succ}^{C}$ for the set of candidates $\mathscr{C} = \{a, b, c, d\}$ and the ranking $a \succ b \succ c \succ d$ is shown.

Note that the definition of consistency is equivalent to the instance of the representation of votes induced by the profile being decreasing on $\sqsupset_{\succ}^{C}$.

Proposition 1. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$ and $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be a representation of votes. The instance $T_{\mathscr{R}} : \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ of T induced by the profile $\mathscr{R}$ is*

(i) *consistent w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ (with corresponding $\sqsupset_{\succ}^{C}$) if and only if, for any $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}_{\neq}^2$, it holds that*

$$(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{C} (a_{j_1}, a_{j_2}) \Rightarrow T(a_{i_1}, a_{i_2}) \rhd_T T(a_{j_1}, a_{j_2})\,;$$

(ii) *weakly consistent w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ (with corresponding $\sqsupset_{\succ}^{C}$) if and only if, for any $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}_{\neq}^2$, it holds that*

$$(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{C} (a_{j_1}, a_{j_2}) \Rightarrow T(a_{i_1}, a_{i_2}) \trianglerighteq_T T(a_{j_1}, a_{j_2})\,.$$

Proof. The result follows from the fact that $(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{C} (a_{j_1}, a_{j_2})$ is equivalent to $a_{i_1} \succ a_{i_2}$ and $(a_{j_1}, a_{j_2}) = (a_{i_2}, a_{i_1})$. □

A stronger condition than that of consistency is monotonicity. Monotonicity w.r.t. a ranking $\succ$ on $\mathscr{C}$ means that, if $a_{i_1} \succ a_{i_2}$, then the support of a_{i_1} over a_{i_2} needs to be greater than (or equal to) the support of a_{i_2} over a_{i_1} and, at the same time, for any third candidate a_{i_3}, the support of a_{i_1} over a_{i_3} needs to be greater than or equal to the support of a_{i_2} over a_{i_3}.[7]

[7] In order to illustrate the definition of monotonicity, we do not mention that the support of a_{i_3} over a_{i_2} also needs to be greater than or equal to the support of a_{i_3} over a_{i_1}.

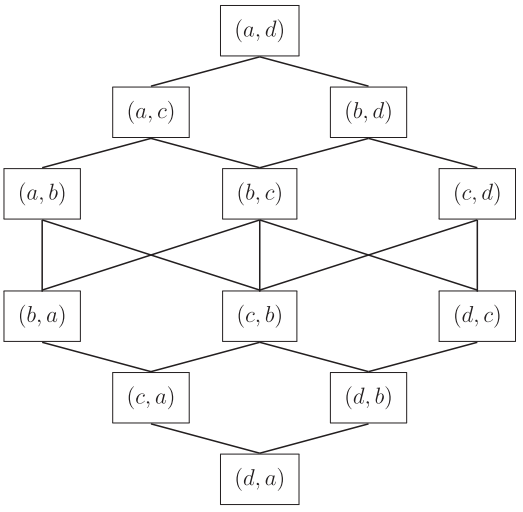

Fig. 2. Hasse diagram of $\sqsupset_{\succ}^{M}$ for the ranking $a \succ b \succ c \succ d$.

Definition 9. Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$ and $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be a representation of votes. The instance $T_{\mathscr{R}} : \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ of T induced by the profile $\mathscr{R}$ is said to be

(i) (strictly) monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ if, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}_{\neq}^2$, it holds that

$$a_{i_1} \succ a_{i_2} \Rightarrow \begin{cases} T_{\mathscr{R}}(a_{i_1}, a_{i_2}) \rhd_T T_{\mathscr{R}}(a_{i_2}, a_{i_1}), \\ \left(\forall a_{i_3} \in \mathscr{C} \setminus \{a_{i_1}, a_{i_2}\}\right)\left(T_{\mathscr{R}}(a_{i_1}, a_{i_3}) \trianglerighteq_T T_{\mathscr{R}}(a_{i_2}, a_{i_3})\right), \\ \left(\forall a_{i_3} \in \mathscr{C} \setminus \{a_{i_1}, a_{i_2}\}\right)\left(T_{\mathscr{R}}(a_{i_3}, a_{i_2}) \trianglerighteq_T T_{\mathscr{R}}(a_{i_3}, a_{i_1})\right); \end{cases}$$

(ii) weakly monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ if, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}_{\neq}^2$, it holds that

$$a_{i_1} \succ a_{i_2} \Rightarrow \begin{cases} T_{\mathscr{R}}(a_{i_1}, a_{i_2}) \trianglerighteq_T T_{\mathscr{R}}(a_{i_2}, a_{i_1}), \\ \left(\forall a_{i_3} \in \mathscr{C} \setminus \{a_{i_1}, a_{i_2}\}\right)\left(T_{\mathscr{R}}(a_{i_1}, a_{i_3}) \trianglerighteq_T T_{\mathscr{R}}(a_{i_2}, a_{i_3})\right), \\ \left(\forall a_{i_3} \in \mathscr{C} \setminus \{a_{i_1}, a_{i_2}\}\right)\left(T_{\mathscr{R}}(a_{i_3}, a_{i_2}) \trianglerighteq_T T_{\mathscr{R}}(a_{i_3}, a_{i_1})\right). \end{cases}$$

Intuitively, we can see that monotonicity w.r.t. a ranking $\succ$ on $\mathscr{C}$ means that the instance of the representation of votes induced by the profile is decreasing on the diagram defined by the following strict partial order relation [22].

Definition 10. Let $\mathscr{C}$ be a set of k candidates. A ranking $\succ \in \mathcal{L}(\mathscr{C})$ induces a strict partial order relation $\sqsupset_{\succ}^{M}$ on $\mathscr{C}_{\neq}^2$ such that, for any two $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}_{\neq}^2$, it holds that $(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{M} (a_{j_1}, a_{j_2})$ if

$$(a_{i_1} \succeq a_{j_1}) \wedge (a_{j_2} \succeq a_{i_2}) \wedge (a_{i_1} \succ a_{j_1} \vee a_{j_2} \succ a_{i_2}).$$

In Fig. 2, a graphical representation of $\sqsupset_{\succ}^{M}$ for the set of candidates $\mathscr{C} = \{a, b, c, d\}$ and the ranking $a \succ b \succ c \succ d$ is shown.

Note that the definition of monotonicity is equivalent to the instance of the representation of votes induced by the profile being decreasing on $\sqsupset_{\succ}^{M}$.

Proposition 2. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$ and $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be a representation of votes. The instance $T_{\mathscr{R}} : \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ of T induced by the profile $\mathscr{R}$ is*

(i) *monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ (with corresponding $\sqsupset_{\succ}^{M}$) if and only if, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}_{\neq}^2$, it holds that*

$$a_{i_1} \succ a_{i_2} \Rightarrow T(a_{i_1}, a_{i_2}) \rhd_T T(a_{i_2}, a_{i_1}),$$

and, for any $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}_{\neq}^2$, it holds that

$$(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{M} (a_{j_1}, a_{j_2}) \Rightarrow T(a_{i_1}, a_{i_2}) \trianglerighteq_T T(a_{j_1}, a_{j_2});$$

(ii) *weakly monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ (with corresponding $\sqsupset_{\succ}^{M}$) if and only if, for any $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}_{\neq}^2$, it holds that*

$$(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{M} (a_{j_1}, a_{j_2}) \Rightarrow T(a_{i_1}, a_{i_2}) \trianglerighteq_T T(a_{j_1}, a_{j_2}).$$

Proof. We prove that statement (ii) holds (statement (i) follows directly from statement (ii)).

The implication $\Rightarrow$. Let us suppose that $T_{\mathscr{R}}$ is weakly monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$. If $(a_{i_1}, a_{i_2}) \sqsupset^M_{\succ} (a_{j_1}, a_{j_2})$, then, by definition of $\sqsupset^M_{\succ}$, it holds that

$$(a_{i_1} \succeq a_{j_1}) \wedge (a_{j_2} \succeq a_{i_2}) \wedge (a_{i_1} \succ a_{j_1} \vee a_{j_2} \succ a_{i_2}) .$$

We need to distinguish three cases: ($a_{i_1} \succ a_{j_1}$ and $a_{j_2} \succ a_{i_2}$), ($a_{i_1} = a_{j_1}$ and $a_{j_2} \succ a_{i_2}$) and ($a_{i_1} \succ a_{j_1}$ and $a_{j_2} = a_{i_2}$).

1. If $a_{i_1} \succ a_{j_1}$ and $a_{j_2} \succ a_{i_2}$, then, by definition of weak monotonicity w.r.t. $\succ$, it holds that:
$$T(a_{i_1}, a_{i_2}) \trianglerighteq_T T(a_{j_1}, a_{i_2}) \trianglerighteq_T T(a_{j_1}, a_{j_2}) ,$$
in case $a_{i_2} \neq a_{j_1}$, or that
$$T(a_{i_1}, a_{i_2}) \trianglerighteq_T T(a_{i_1}, a_{j_2}) \trianglerighteq_T T(a_{j_1}, a_{j_2}) ,$$
in case $a_{i_1} \neq a_{j_2}$, or that
$$T(a_{i_1}, a_{i_2}) \trianglerighteq_T T(a_{i_2}, a_{i_1}) = T(a_{j_1}, a_{j_2}) ,$$
in case $a_{i_2} = a_{j_1}$ and $a_{i_1} = a_{j_2}$.
2. If $a_{i_1} = a_{j_1}$ and $a_{j_2} \succ a_{i_2}$, then, by definition of weak monotonicity w.r.t. $\succ$, it holds that:
$$T(a_{i_1}, a_{i_2}) = T(a_{j_1}, a_{i_2}) \trianglerighteq_T T(a_{j_1}, a_{j_2}) .$$
3. If $a_{i_1} \succ a_{j_1}$ and $a_{j_2} = a_{i_2}$, then, by definition of weak monotonicity w.r.t. $\succ$, it holds that:
$$T(a_{i_1}, a_{i_2}) \trianglerighteq_T T(a_{j_1}, a_{i_2}) = T(a_{j_1}, a_{j_2}) .$$

The implication $\Leftarrow$. This result follows directly from the fact that, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$ such that $a_{i_1} \succ a_{i_2}$ and any $a_{i_3} \in \mathscr{C} \backslash \{a_{i_1}, a_{i_2}\}$, it holds that $(a_{i_1}, a_{i_2}) \sqsupset^M_{\succ} (a_{i_2}, a_{i_1})$, $(a_{i_1}, a_{i_3}) \sqsupset^M_{\succ} (a_{i_2}, a_{i_3})$ and $(a_{i_3}, a_{i_2}) \sqsupset^M_{\succ} (a_{i_3}, a_{i_1})$. □

The notion of pseudo-monotonicity[8] introduced in [19] lies in between consistency and monotonicity. This property means that, if $a_{i_1} \succ a_{i_2} \succ a_{i_3}$, then not only the support of a_{i_1} over a_{i_3} needs to be greater than (or equal to) the support of a_{i_3} over a_{i_1} but it also needs to be greater than or equal to both the support of a_{i_1} over a_{i_2} and the support of a_{i_2} over a_{i_3}.[9]

Definition 11. Let $\mathscr{C}$ be a set of k candidates. A ranking $\succ \in \mathcal{L}(\mathscr{C})$ induces a strict partial order relation $\sqsupset^P_{\succ}$ on $\mathscr{C}^2_{\neq}$ such that, for any two *different* $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}^2_{\neq}$, it holds that $(a_{i_1}, a_{i_2}) \sqsupset^P_{\succ} (a_{j_1}, a_{j_2})$ if one of the following conditions holds:

(i) $a_{i_1} \succeq a_{j_1} \succ a_{j_2} \succeq a_{i_2}$;
(ii) $a_{j_2} \succeq a_{i_2} \succ a_{i_1} \succeq a_{j_1}$;
(iii) $\big(\exists \ell \in \{1, \ldots, k-1\}\big)\big((a_{i_1} \succeq a_{\ell} \succ a_{\ell+1} \succeq a_{i_2}) \wedge (a_{j_2} \succeq a_{\ell} \succ a_{\ell+1} \succeq a_{j_1})\big)$.

In Fig. 3, a graphical representation of $\sqsupset^P_{\succ}$ for the set of candidates $\mathscr{C} = \{a, b, c, d\}$ and the ranking $a \succ b \succ c \succ d$ is shown. Note that condition (i) corresponds to the case where both couples of candidates are in the upper half of the diagram; condition (ii) corresponds to the case where both couples of candidates are in the lower half of the diagram; condition (iii) corresponds to the case where the first couple of candidates is in the upper half and the second couple of candidates is in the lower half.

Definition 12. Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$ and $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}^2_{\neq} \to \mathbb{R}^n$ be a representation of votes. The instance $T_{\mathscr{R}} : \mathscr{C}^2_{\neq} \to \mathbb{R}^n$ of T induced by the profile $\mathscr{R}$ is said to be

(i) (strictly) pseudo-monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ (with corresponding $\sqsupset^P_{\succ}$) if, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$, it holds that
$$a_{i_1} \succ a_{i_2} \Rightarrow T_{\mathscr{R}}(a_{i_1}, a_{i_2}) \triangleright_T T_{\mathscr{R}}(a_{i_2}, a_{i_1}) ,$$
and, for any $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}^2_{\neq}$, it holds that
$$(a_{i_1}, a_{i_2}) \sqsupset^P_{\succ} (a_{j_1}, a_{j_2}) \Rightarrow T_{\mathscr{R}}(a_{i_1}, a_{i_2}) \trianglerighteq_T T_{\mathscr{R}}(a_{j_1}, a_{j_2}) ;$$

(ii) weakly pseudo-monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$ (with corresponding $\sqsupset^P_{\succ}$) if, for any $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}^2_{\neq}$, it holds that
$$(a_{i_1}, a_{i_2}) \sqsupset^P_{\succ} (a_{j_1}, a_{j_2}) \Rightarrow T_{\mathscr{R}}(a_{i_1}, a_{i_2}) \trianglerighteq_T T_{\mathscr{R}}(a_{j_1}, a_{j_2}) .$$

[8] Pseudo-monotonicity actually is referred to as monotonicity in [19].

[9] In order to illustrate the definition of pseudo-monotonicity, we do not mention that the support of a_{i_1} over a_{i_2} also needs to be greater than (or equal) to the support of a_{i_2} over a_{i_1}; that the support of a_{i_2} over a_{i_3} also needs to be greater than (or equal) to the support of a_{i_3} over a_{i_2}; and that both the support of a_{i_2} over a_{i_1} and the support of a_{i_3} over a_{i_2} also need to be greater than or equal to the support of a_{i_3} over a_{i_1}.

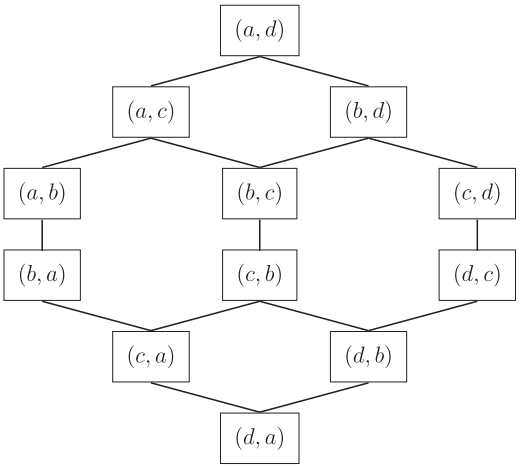

Fig. 3. Hasse diagram of $\sqsupset_{\succ}^{P}$ for the ranking $a \succ b \succ c \succ d$.

A further analysis of the diagrams corresponding to monotonicity and pseudo-monotonicity shows that pseudo-monotonicity is not actually linked with a ranking $\succ$ on the set of candidates but on its antitransitive part, i.e.

$$\succ' = \{(a_{i_1}, a_{i_2}) \in \mathscr{C}^2 \mid (a_{i_1} \succ a_{i_2}) \wedge (\nexists a_{i_3} \in \mathscr{C})(a_{i_1} \succ a_{i_3} \succ a_{i_2})\}.$$

Therefore, although both monotonicity and pseudo-monotonicity are equivalent for some representation of votes (for instance all M^{α}-majority matrices and the votrix), we advocate to consider from now on monotonicity instead of pseudo-monotonicity.

As $\trianglerighteq_T$ is defined as the reflexive closure of $\triangleright_T$, it is straightforward to prove that consistency, pseudo-monotonicity and monotonicity respectively imply weak consistency, weak pseudo-monotonicity and weak monotonicity.

Proposition 3. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$, $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be a representation of votes and $T_{\mathscr{R}} : \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be the instance of T induced by the profile $\mathscr{R}$.*

(i) *If $T_{\mathscr{R}}$ is consistent w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$, then $T_{\mathscr{R}}$ is weakly consistent w.r.t. $\succ$.*
(ii) *If $T_{\mathscr{R}}$ is pseudo-monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$, then $T_{\mathscr{R}}$ is weakly pseudo-monotone w.r.t. $\succ$.*
(iii) *If $T_{\mathscr{R}}$ is monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$, then $T_{\mathscr{R}}$ is weakly monotone w.r.t. $\succ$.*

Furthermore, in case the instance of the representation of votes induced by the profile is consistent w.r.t. a ranking on the set of candidates, both monotonicity and weak monotonicity coincide. The result also holds for pseudo-monotonicity and weak pseudo-monotonicity.

Proposition 4. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$, $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be a representation of votes and $T_{\mathscr{R}} : \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be the instance of T induced by the profile $\mathscr{R}$.*

1. *$T_{\mathscr{R}}$ is monotone w.r.t. a ranking $\succ$ if and only if $T_{\mathscr{R}}$ is consistent and weakly monotone w.r.t. $\succ$.*
2. *$T_{\mathscr{R}}$ is pseudo-monotone w.r.t. a ranking $\succ$ if and only if $T_{\mathscr{R}}$ is consistent and weakly pseudo-monotone w.r.t. $\succ$.*

Proof. The result trivially follows from the corresponding definitions and the fact that $\triangleright_T \subseteq \trianglerighteq_T$. □

An important topic of study is the relation between the monotonicity, pseudo-monotonicity and consistency properties. Note that monotonicity is the strongest property among the three, consistency is the weakest one and pseudo-monotonicity lies in between monotonicity and consistency.

Theorem 1. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$, $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be a representation of votes and $T_{\mathscr{R}} : \mathscr{C}_{\neq}^2 \to \mathbb{R}^n$ be the instance of T induced by the profile $\mathscr{R}$.*

(i) *If $T_{\mathscr{R}}$ is monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$, then $T_{\mathscr{R}}$ is pseudo-monotone w.r.t. $\succ$.*
(ii) *If $T_{\mathscr{R}}$ is pseudo-monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$, then $T_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*

Proof. First, we prove that $\sqsupset_{\succ}^{P} \subseteq \sqsupset_{\succ}^{M}$. For any two different $(a_{i_1}, a_{i_2}), (a_{j_1}, a_{j_2}) \in \mathscr{C}_{\neq}^2$ such that $(a_{i_1}, a_{i_2}) \sqsupset_{\succ}^{P} (a_{j_1}, a_{j_2})$, it holds that $a_{i_1} \succeq a_{j_1} \succ a_{j_2} \succeq a_{i_2}$ or $a_{j_2} \succeq a_{i_2} \succ a_{i_1} \succeq a_{j_1}$ or there exists $\ell \in \{1, \ldots, k-1\}$ such that $a_{i_1} \succeq a_{\ell} \succ a_{\ell+1} \succeq a_{i_2}$ and $a_{j_2} \succeq a_{\ell} \succ a_{\ell+1} \succeq a_{j_1}$. In all three cases, it holds that

$$(a_{i_1} \succeq a_{j_1}) \wedge (a_{j_2} \succeq a_{i_2}).$$

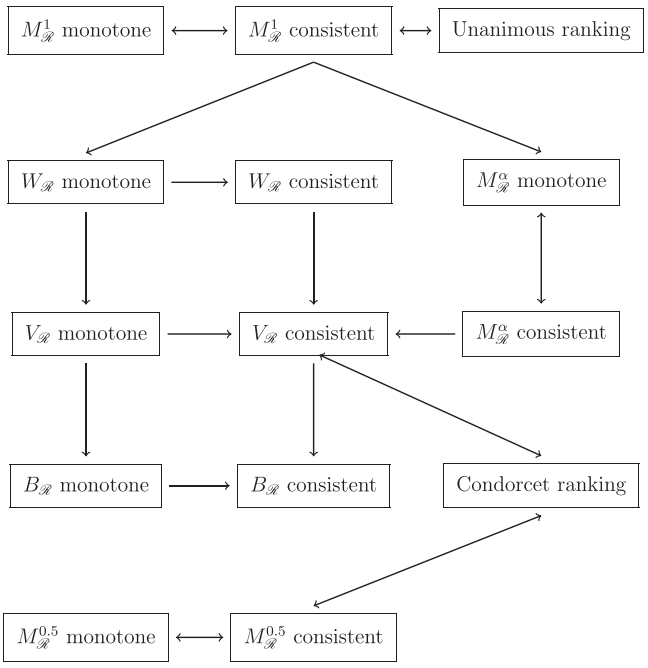

Fig. 4. Comparison between different properties of the representations of votes.

In addition, as $(a_{i_1}, a_{i_2}) \neq (a_{j_1}, a_{j_2})$, one of the following holds:

$$(a_{i_1} \succ a_{j_1}) \vee (a_{j_2} \succ a_{i_2}).$$

We conclude that $(a_{i_1}, a_{i_2}) \sqsupset^M_\succ (a_{j_1}, a_{j_2})$ and, thus, $\sqsupset^P_\succ \subseteq \sqsupset^M_\succ$. Therefore, statement (i) follows from the characterization of monotonicity given in Proposition 2 and the fact that $\sqsupset^P_\succ \subseteq \sqsupset^M_\succ$.

Statement (ii) follows immediately from the definition of pseudo-monotonicity. □

Corollary 1. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $n \in \mathbb{N}$, $T : \mathcal{L}(\mathscr{C})^r \times \mathscr{C}^2_{\neq} \to \mathbb{R}^n$ be a representation of votes and $T_{\mathscr{R}} : \mathscr{C}^2_{\neq} \to \mathbb{R}^n$ be the instance of T induced by the profile $\mathscr{R}$.*

(i) *If $T_{\mathscr{R}}$ is weakly monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$, then $T_{\mathscr{R}}$ is weakly pseudo-monotone w.r.t. $\succ$.*
(ii) *If $T_{\mathscr{R}}$ is weakly pseudo-monotone w.r.t. a ranking $\succ \in \mathcal{L}(\mathscr{C})$, then $T_{\mathscr{R}}$ is weakly consistent w.r.t. $\succ$.*

4. General comparison

The relation between the properties of consistency and monotonicity of all different representations of votes introduced in Section 2 is a relevant topic of study. Throughout this section, we will prove that the implications shown in Fig. 4 are satisfied.

First, we prove that, for any $\alpha \in [\frac{1}{2}, 1]$, consistency and monotonicity of the α-majority matrix induced by the profile $\mathscr{R}$ are equivalent.

Theorem 2. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $\alpha \in [\frac{1}{2}, 1]$ and $M^\alpha_{\mathscr{R}}$ be the α-majority matrix induced by the profile $\mathscr{R}$. $M^\alpha_{\mathscr{R}}$ is monotone w.r.t. a ranking $\succ$ on $\mathscr{C}$ if and only if $M^\alpha_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*

Proof. Note that the fact that $M^\alpha_{\mathscr{R}}$ is monotone w.r.t. a ranking $\succ$ on $\mathscr{C}$ already implies that $M^\alpha_{\mathscr{R}}$ is consistent w.r.t. $\succ$ (Theorem 1). Let us prove that the fact that $M^\alpha_{\mathscr{R}}$ is consistent w.r.t. a ranking $\succ$ on $\mathscr{C}$ implies that $M^\alpha_{\mathscr{R}}$ is monotone w.r.t. $\succ$.

If $M^\alpha_{\mathscr{R}}$ is consistent w.r.t. a ranking $\succ$ on $\mathscr{C}$, then, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that $M^\alpha_{\mathscr{R}}(a_{i_1}, a_{i_2}) > M^\alpha_{\mathscr{R}}(a_{i_2}, a_{i_1})$. As the α-majority matrix takes values in {0, 1}, it implies that, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that $M^\alpha_{\mathscr{R}}(a_{i_1}, a_{i_2}) = 1$ and $M^\alpha_{\mathscr{R}}(a_{i_2}, a_{i_1}) = 0$.

For any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$ and any $a_{i_3} \in \mathscr{C} \backslash \{a_{i_1}, a_{i_2}\}$, we distinguish three cases: $a_{i_3} \succ a_{i_1} \succ a_{i_2}$, $a_{i_1} \succ a_{i_3} \succ a_{i_2}$ and $a_{i_1} \succ a_{i_2} \succ a_{i_3}$.

1. If $a_{i_3} \succ a_{i_1} \succ a_{i_2}$, then it holds that $M^\alpha_{\mathscr{R}}(a_{i_1}, a_{i_3}) = 0$, $M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_1}) = 1$, $M^\alpha_{\mathscr{R}}(a_{i_2}, a_{i_3}) = 0$ and $M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_2}) = 1$.
2. If $a_{i_1} \succ a_{i_3} \succ a_{i_2}$, then it holds that $M^\alpha_{\mathscr{R}}(a_{i_1}, a_{i_3}) = 1$, $M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_1}) = 0$, $M^\alpha_{\mathscr{R}}(a_{i_2}, a_{i_3}) = 0$ and $M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_2}) = 1$.
3. If $a_{i_1} \succ a_{i_2} \succ a_{i_3}$, then it holds that $M^\alpha_{\mathscr{R}}(a_{i_1}, a_{i_3}) = 1$, $M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_1}) = 0$, $M^\alpha_{\mathscr{R}}(a_{i_2}, a_{i_3}) = 1$ and $M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_2}) = 0$.

We conclude that $M^\alpha_{\mathscr{R}}(a_{i_1}, a_{i_3}) \geq M^\alpha_{\mathscr{R}}(a_{i_2}, a_{i_3})$ and $M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_2}) \geq M^\alpha_{\mathscr{R}}(a_{i_3}, a_{i_1})$. Thus, $M^\alpha_{\mathscr{R}}$ is monotone w.r.t. $\succ$. □

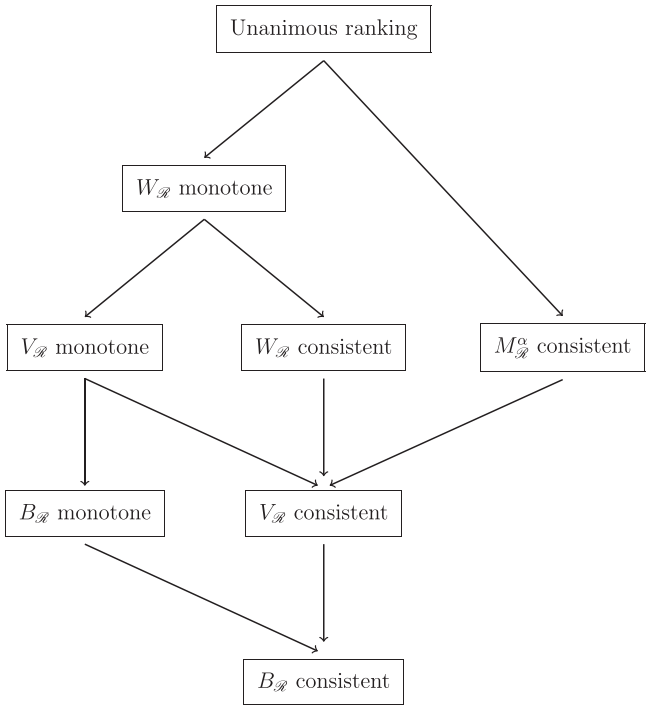

Fig. 5. Comparison between different properties of the representations of votes (reduced).

By definition of the unanimous ranking[10] and the Condorcet ranking[11] and as a consequence of the previous theorem, the following two propositions are straightforward.

Proposition 5. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $M^1_{\mathscr{R}}$ be the 1-majority matrix induced by the profile $\mathscr{R}$ and $\succ$ be a ranking on $\mathscr{C}$. The following three statements are equivalent:*

(i) *$M^1_{\mathscr{R}}$ is monotone w.r.t. $\succ$.*
(ii) *$M^1_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*
(iii) *$\succ$ is the unanimous ranking.*

Proposition 6. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $M^{0.5}_{\mathscr{R}}$ be the 0.5-majority matrix induced by the profile $\mathscr{R}$, $V_{\mathscr{R}}$ be the votrix induced by the profile $\mathscr{R}$ and $\succ$ be a ranking on $\mathscr{C}$. The following four statements are equivalent:*

(i) *$M^{0.5}_{\mathscr{R}}$ is monotone w.r.t. $\succ$.*
(ii) *$M^{0.5}_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*
(iii) *$V_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*
(iv) *$\succ$ is the Condorcet ranking.*

Due to the previous results, the implications shown in Fig. 4 can be reduced to the implications shown in Fig. 5.
The consistency of the α-majority matrix implies the consistency of all the β-majority matrices with $\alpha > \beta$.

Proposition 7. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $\alpha, \beta \in [\frac{1}{2}, 1]$ such that $\alpha > \beta$ and $M^{\alpha}_{\mathscr{R}}$ and $M^{\beta}_{\mathscr{R}}$ be, respectively, the α-majority matrix and the β-majority matrix induced by the profile $\mathscr{R}$. If $M^{\alpha}_{\mathscr{R}}$ is consistent w.r.t. a ranking $\succ$ on $\mathscr{C}$, then $M^{\beta}_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*

Proof. If $M^{\alpha}_{\mathscr{R}}$ is consistent w.r.t. a ranking $\succ$ on $\mathscr{C}$, then, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that $M^{\alpha}_{\mathscr{R}}(a_{i_1}, a_{i_2}) > M^{\alpha}_{\mathscr{R}}(a_{i_2}, a_{i_1})$. As the α-majority matrix only takes values in {0, 1}, it holds that $M^{\alpha}_{\mathscr{R}}(a_{i_1}, a_{i_2}) = 1$ and $M^{\alpha}_{\mathscr{R}}(a_{i_2}, a_{i_1}) = 0$. There-

[10] For a profile of rankings $\mathscr{R}$, a ranking $\succ$ is called the unanimous ranking if all the rankings in $\mathscr{R}$ coincide with $\succ$.
[11] For a profile of rankings $\mathscr{R}$, a ranking $\succ$ is called the Condorcet ranking if every candidate is preferred by more than half of the voters to all the candidates ranked after him/her.

fore, it holds that

$$\#\{j \in \{1,\dots,r\} \mid a_{i_1} \succ_j a_{i_2}\} \geq r \cdot \alpha > r \cdot \beta\,,$$
$$\#\{j \in \{1,\dots,r\} \mid a_{i_2} \succ_j a_{i_1}\} \leq r - r \cdot \alpha < \frac{r}{2} \leq r \cdot \beta\,.$$

We conclude that, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that $M^{\beta}_{\mathscr{R}}(a_{i_1}, a_{i_2}) = 1 > 0 = M^{\beta}_{\mathscr{R}}(a_{i_2}, a_{i_1})$. □

In the following theorem, we analyse the relation between the monotonicity of the votex, the monotonicity of the votrix and the monotonicity of the beatpath matrix. We will prove that monotonicity of the votex is a stronger condition than both monotonicity of the votrix and monotonicity of the beatpath matrix. Furthermore, monotonicity of the votrix is a stronger condition than monotonicity of the beatpath matrix.

Theorem 3. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $B_{\mathscr{R}}$ be the beatpath matrix induced by the profile $\mathscr{R}$, $V_{\mathscr{R}}$ be the votrix induced by the profile $\mathscr{R}$, $W_{\mathscr{R}}$ be the votex induced by the profile $\mathscr{R}$ and $\succ$ be a ranking on $\mathscr{C}$. The following three statements hold:*

(i) *If $\succ$ is the unanimous ranking, then $W_{\mathscr{R}}$ is monotone w.r.t. $\succ$.*
(ii) *If $W_{\mathscr{R}}$ is monotone w.r.t.$\succ$, then $V_{\mathscr{R}}$ is monotone w.r.t. $\succ$.*
(iii) *If $V_{\mathscr{R}}$ is monotone w.r.t.$\succ$, then $B_{\mathscr{R}}$ is monotone w.r.t. $\succ$.*

Proof.

(i) If $\succ$ is the unanimous ranking, then, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$, $W_{\mathscr{R}}(a_{i_1}, a_{i_2})$ is the vector where the $(i_2 - i_1)$th component equals r and all the other components are equal to zero. It is immediate to see that W is monotone w.r.t. $\succ$.

(ii) By definition of $\rhd_W$ and from the fact that

$$\#\{j \in \{1,\dots,r\} \mid a_{i_1} \succ_j a_{i_2}\} = \sum_{i=1}^{k-1} \#\{j \in \{1,\dots,r\} \mid P_j(a_{i_2}) - P_j(a_{i_1}) = i\}\,,$$

it follows that $V_{\mathscr{R}}$ is monotone w.r.t. $\succ$.

(iii) In order to ease the proof of this statement, and just throughout this proof, we consider the notation $r_{\succ}(i)$ to refer to the label corresponding to the candidate ranked at the ith position in $\succ$. We first prove that, if $V_{\mathscr{R}}$ is monotone w.r.t. $\succ$, then the beatpath from a candidate a_{i_1} to a different candidate a_{i_2} is the couple (a_{i_1}, a_{i_2}) in case $a_{i_1} \succ a_{i_2}$ and the list of couples $(a_\ell, a_{\ell+1})_{\ell=r_{\succ}(i_1)}^{r_{\succ}(i_2)-1}$ in case $a_{i_2} \succ a_{i_1}$.

1. The case $a_{i_1} \succ a_{i_2}$. Suppose that there is a path from a_{i_1} to a_{i_2} such that the strength of its weakest link is greater than the strength of support of a_{i_1} over a_{i_2}. We distinguish three cases:
   (a) There is a candidate a_{i_3} in the path such that $a_{i_3} \succ a_{i_1} \succ a_{i_2}$. Therefore, one of the couples in the path is such that the first element is preferred to the second one by less than half of the number of voters, a contradiction with the fact that the strength of the weakest link is greater than the strength of support of a_{i_1} over a_{i_2}.
   (b) There is a candidate a_{i_3} in the path such that $a_{i_1} \succ a_{i_2} \succ a_{i_3}$. Therefore, one of the couples in the path is such that the first element is preferred to the second one by less than half of the number of voters, a contradiction with the fact that the strength of the weakest link is greater than the strength of support of a_{i_1} over a_{i_2}.
   (c) Any candidate a_{i_3} in the path is such that $a_{i_1} \succ a_{i_3} \succ a_{i_2}$. Therefore, as $V_{\mathscr{R}}$ is monotone w.r.t. $\succ$, the strength of support of a_{i_1} over a_{i_2} is greater than or equal to the strength of all the links in the path, a contradiction with the fact that the strength of the weakest link is greater than the strength of support of a_{i_1} over a_{i_2}.
2. The case $a_{i_2} \succ a_{i_1}$. Suppose that there is a path from a_{i_2} to a_{i_1} such that the strength of its weakest link is greater than the lowest strength of support of a_ℓ over $a_{\ell+1}$, for $\ell \in \{r_{\succ}(i_1), \dots, r_{\succ}(i_2) - 1\}$. We prove that, for any $(a_\ell, a_{\ell+1})$ with $\ell \in \{r_{\succ}(i_1), \dots, r_{\succ}(i_2) - 1\}$, there is a couple (a_{j_1}, a_{j_2}) in the path such that $(a_\ell, a_{\ell+1}) \sqsupset^M_{\succ} (a_{j_1}, a_{j_2})$ or $(a_\ell, a_{\ell+1}) = (a_{j_1}, a_{j_2})$. Let us suppose that, for a fixed $\ell \in \{r_{\succ}(i_1), \dots, r_{\succ}(i_2) - 1\}$, there is no (a_{j_1}, a_{j_2}) in the path such that $(a_\ell, a_{\ell+1}) \sqsupset^M_{\succ} (a_{j_1}, a_{j_2})$ or $(a_\ell, a_{\ell+1}) = (a_{j_1}, a_{j_2})$. It follows that there is no (a_{j_1}, a_{j_2}) in the path such that $a_\ell \succeq a_{j_1}$ or that $a_{j_2} \succeq a_{\ell+1}$ or that $(a_\ell, a_{\ell+1}) = (a_{j_1}, a_{j_2})$. However, as $a_{i_2} \succeq a_{\ell+1} \succ a_\ell \succeq a_{i_1}$, there needs to be a couple (a_{j_1}, a_{j_2}) in the path such that $a_{j_2} \succeq a_{\ell+1} \succ a_\ell \succeq a_{j_1}$ due to the transitivity of $\succ$, the definition of path and the fact that a_ℓ and $a_{\ell+1}$ are consecutive candidates in $\succ$. We conclude that $a_\ell \succeq a_{j_1}$ or that $a_{j_2} \succeq a_{\ell+1}$ or that $(a_\ell, a_{\ell+1}) = (a_{j_1}, a_{j_2})$, a contradiction.
   Finally, as $V_{\mathscr{R}}$ is monotone w.r.t. $\succ$, the strength of the weakest link in the path is lower than or equal to the strength of the lowest strength of support of a_ℓ over $a_{\ell+1}$, for $\ell \in \{r_{\succ}(i_1), \dots, r_{\succ}(i_2) - 1\}$, a contradiction.

We conclude that, in case $V_{\mathscr{R}}$ is monotone w.r.t. the ranking $\succ$, the beatpath matrix $B_{\mathscr{R}}$ equals $V_{\mathscr{R}}$ for couples in the upper half of the diagram of $\sqsupset^M_{\succ}$ and that for any couple in the lower half of the diagram it equals the minimum of the values of $V_{\mathscr{R}}$ in the upper row of the lower half of the diagram that dominate the chosen couple. Thus, as $V_{\mathscr{R}}$ is monotone w.r.t. $\succ$, $B_{\mathscr{R}}$ obviously is decreasing on $\sqsupset^M_{\succ}$ (weak monotonicity w.r.t. $\succ$ holds). In addition, as the strength of support of any couple in the upper half of the diagram is greater than the strength of support of any couple in the lower half of the diagram, consistency w.r.t. $\succ$ holds. By Proposition 4 and due to the fact that $B_{\mathscr{R}}$ is weakly monotone and consistent w.r.t. $\succ$, we conclude that $B_{\mathscr{R}}$ is monotone w.r.t. $\succ$. □

In the same way, consistency of the votex is a stronger condition than both consistency of the votrix and consistency of the beatpath matrix. Furthermore, consistency of the votrix is a stronger condition than consistency of the beatpath matrix.

Theorem 4. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings given by the voters, $B_{\mathscr{R}}$ be the beatpath matrix induced by the profile $\mathscr{R}$, $V_{\mathscr{R}}$ be the votrix induced by the profile $\mathscr{R}$, $W_{\mathscr{R}}$ be the votex induced by the profile $\mathscr{R}$. The following three statements hold:*

(i) *If $\succ$ is the unanimous ranking, then $W_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*
(ii) *If $W_{\mathscr{R}}$ is consistent w.r.t. a ranking $\succ$ on $\mathscr{C}$, then $V_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*
(iii) *If $V_{\mathscr{R}}$ is consistent w.r.t. a ranking $\succ$ on $\mathscr{C}$, then $B_{\mathscr{R}}$ is consistent w.r.t. $\succ$.*

Proof.

(i) It trivially follows from statement (i) of Theorem 3 and from the fact that monotonicity of a representation of votes w.r.t. a ranking $\succ$ on $\mathscr{C}$ implies consistency of the representation of votes w.r.t. $\succ$, proved in Theorem 1.

(ii) From the fact that

$$\#\{j \in \{1,\dots,r\} \mid a_{i_1} \succ_j a_{i_2}\} = \sum_{i=1}^{k-1} \#\{j \in \{1,\dots,r\} \mid P_j(a_{i_2}) - P_j(a_{i_1}) = i\},$$

we conclude that, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$, $\sum_{i=1}^{k-1} W_{\mathscr{R}}(a_{i_1}, a_{i_2})(i) = V_{\mathscr{R}}(a_{i_1}, a_{i_2})$.
As $W_{\mathscr{R}}$ is consistent w.r.t. $\succ$, it holds that, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$ such that $a_{i_1} \succ a_{i_2}$,

$$V_{\mathscr{R}}(a_{i_1}, a_{i_2}) = \sum_{i=1-k}^{-1} W_{\mathscr{R}}(a_{i_1}, a_{i_2})(i) > \sum_{i=1-k}^{-1} W_{\mathscr{R}}(a_{i_2}, a_{i_1})(i) = V_{\mathscr{R}}(a_{i_2}, a_{i_1}).$$

Therefore, $V_{\mathscr{R}}$ is consistent w.r.t. $\succ$.

(iii) Note that $B_{\mathscr{R}}(a_{i_1}, a_{i_2}) \geq V_{\mathscr{R}}(a_{i_1}, a_{i_2})$ for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$. Therefore, as $V_{\mathscr{R}}$ is consistent w.r.t. $\succ$, it holds that for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$ such that $a_{i_1} \succ a_{i_2}$,

$$B_{\mathscr{R}}(a_{i_1}, a_{i_2}) \geq V_{\mathscr{R}}(a_{i_1}, a_{i_2}) > \frac{r}{2}.$$

In addition, for any $(a_{i_2}, a_{i_1}) \in \mathscr{C}^2_{\neq}$ such that $a_{i_1} \succ a_{i_2}$, we have that any path going from a_{i_2} to a_{i_1} contains a couple (a_{j_1}, a_{j_2}) such that $a_{j_2} \succ a_{j_1}$. It means that the strength of the weakest link of any path going from a_{i_2} to a_{i_1} is lower than $\frac{r}{2}$. Thus, if $a_{i_1} \succ a_{i_2}$, then it holds that

$$B_{\mathscr{R}}(a_{i_1}, a_{i_2}) > \frac{r}{2} > B_{\mathscr{R}}(a_{i_2}, a_{i_1}).$$

We conclude that $B_{\mathscr{R}}$ is consistent w.r.t. $\succ$.

□

5. Conclusions and open problems

In this work, we have recalled the most relevant representations of votes based on pairwise information. Three properties of these representations of votes have been analysed: consistency, pseudo-monotonicity and monotonicity. As pseudo-monotonicity is linked to the antitransitive part of the ranking, the use of monotonicity over pseudo-monotonicity has been encouraged. The paper has been concluded with an analysis of the relation between consistency and monotonicity of all the representations of votes that have been here recalled.

Pérez-Fernández et al. [21] described the monometric rationalisation of ranking rules as the search for a consensus state by means of an appropriate monometric. A consensus state can be understood as a set of profiles of rankings for which determining the winning ranking is obvious. Consistency of the votrix [4] and of the 1-majority matrix [14] have been described as interesting consensus states in the aggregation of rankings. In a recent paper, we proposed the use of monotonicity of the votrix and of the votex [19] due to the fact that both consensus states imply (in addition to the existence of the Condorcet ranking [4]) the existence of the Borda ranking [2]. Unfortunately, the choice of the most suitable consensus state is unclear and yields no absolute truth. We encourage the use and analysis of the consistency and the monotonicity of the other representations of votes described in this paper. They may lead to natural consensus states and to the introduction of new natural ranking rules. Obviously, the choice of consensus state should always be addressed beforehand, preventing the decision maker to influence the outcome of the voting process.

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